UNIVERSITÉ DE MONTRÉAL

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CRT-59[illegible]

# COMPETITIVE SPATIAL MODELS*

by

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* This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (grants A9160 and A4747).

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Université de Montréal
Centre de recherche sur les transports - Publication #598
October 1988

Dépôt légal - 4e trimestre 1988

## ABSTRACT

This paper presents a survey of competitive spatial models. The major components of this field are first described. The original Hotelling model is then introduced and some of its shortcomings are discussed. Then follow some extensions of the basic model. The case of facilities with equal prices and costs is studied and some results of market share maximization in a simple space are presented. Finally, profit and market share models in other spaces are discussed.



Key words : competitive spatial models, competitive location.


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## RÉSUMÉ

Cet article passe en revue les modèles de compétition spatiale. On décrit d'abord les principales composantes de ce domaine d'étude. Par la suite, on présente le modèle de Hotelling, certaines de ses limites et des extensions du modèle de base. Le cas d'établissements avec prix et coûts égaux est étudié et certains résultats sur la maximisation de la part de marché dans un espace simple sont présentés. On discute finalement des modèles de profit et de part de marché dans des espaces plus complexes.



Mots-clés : modèles de compétition spatiale, localisation compétitive.

## 1. INTRODUCTION

The history of competitive spatial modeling dates back to the seminal paper by Hotelling (1929). Interestingly enough, the last ten years have shown an increased amount of work not only in spatial modeling in general but on the original model in particular. Traditionally, most papers dealing with spatial models are found in geographical and economics journals. Recently, however, these models have proliferated into areas and journals dealing with marketing, politics, and operations research, just to name a few. There are two major reasons for this growth. The first is that analyses of competitive spatial models today involve using tools from fields as diverse as computational geometry, game theory, global optimization, and stochastic processes. We will point out the uses of methods from these fields in our discussion of the individual models below. The second reason is that more and more real-life situations are being described as spatial models. In addition to the more standard physical location of facilities, some popular applications are the location of products in a "feature space" whose dimensions denote some key features of the product under consideration. Another example is the location of political candidates in an "issue space" with the dimensions representing important political issues. Yet another application is found in the area of work allocation where, based on aptitude tests, employees are positioned in "ability spaces" whose dimensions represent some important skills. The goals in the first two examples would be the maximization of the number of customers or voters who, through their preferred points, i.e. preferred product features or favored political views, are also mapped into the appropriate spaces. In the work load allocation model, the objective would be to assign tasks to employees with sufficient

skills to perform the jobs while maintaining a reasonably balanced work load among the employees. For a detailed discussion of spatial modeling the reader is referred to Eiselt [12].

The study of competitive spatial models ties together various strands of research, all dealing with models of the following general form. Given a space and customers distributed in that space, several players are to locate a number of facilities each according to some given rule with some prespecified objectives. Suppose for a moment that all facilities have already been located. Assuming that each customer will then patronize exactly one facility (viz. the one he is most attracted to), it is possible to outline the market areas of the facilities. This is usually done by means of a concept developed by the geographer Thiessen [33] and independently by the Russian mathematician Voronoi [34]. The market areas of the facilities are thus aptly named Voronoi or Thiessen areas. Related work on the propensity of customers to be less attracted to facilities more distant from them has been initiated by Reilly [28] and later by Huff [19] and others. Based on the types of models these researchers are working with, they are frequently referred to as "retail gravitationalists". Once an expression is found by some method which determines the market area for any one facility, we may be interested in locating optimally. The multitude of location models is well known to most operations researchers; recent reviews are those by Domschke and Drexl [10]as well as Love, Morris and Wesolowski [23]. But even once all facilities have been located, the problem is not necessarily solved. Each of the facilities may try to capture more customers and thus increase its profit, market share or whatever its objective may be by changing its price, location, or any other measure

of attractiveness. This dynamic process is governed by the rules of game theory. The application of game theory to spatial models can be thought of as the competitive extension of Löschian [22] models and central place theory. An account of some aspects of competitive location models can be found in Wendell and McKelvey [35].

This paper is organized as follows. In the second section we will outline and discuss the major components of competitive spatial models. In the third section, we describe in some detail the original Hotelling model as well as some of its shortcomings. Some extensions of the model are discussed in the fourth section. In the fifth section we assume equal prices and fixed costs for all facilities so that the objectives of a profit maximizer and that of a market share maximizing facility are identical. Some results of market share maximization in a simple space are presented. Finally, in the sixth section, we elaborate upon profit and market share models in various other spaces.

## 2. COMPONENTS OF THE MODEL

In this section we will outline the major components of competitive spatial models. In particular, we will concentrate on the eight major factors which determine the model. Obviously, many additional factors can be (and have been) included in order to make the model more realistic or to focus on special features.

The first component of the model is that of space. We distinguish between discrete spatial models, mostly on networks, and continuous spatial models in $\mathbb{R}^d$ (with $d \leq 3$ in the case of physical location). A variety of

authors have considered models on a one-dimensional line segment. Clearly, markets of such a simple space are all but nonexistent but even such simplified models may provide some insight into the structure of the problem and the sensitivity of its parameters. The mathematical complexity of the model implies a tradeoff between a simple space with relatively complex additional features and a more complex space with simplified other features of the model. It has been shown in the literature that even simple models in $\mathbb{R}^2$ are highly complicated from a mathematical point of view. In nonphysical location problems, e.g. the location of products in feature spaces or political candidates in issue spaces, modeling in $\mathbb{R}^d$ with $d > 3$ becomes meaningful. Such models tend to be very complex, not only because of the "curse of dimensionality" but also because of the selection of the characteristics (features or issues) and possible autocorrelations between them. Finally, there are the network models, probably the most realistic but frequently plagued by very large scale, i.e. a large number of nodes. By nature, network models tend to involve integer optimization problems whereas the models in $\mathbb{R}^d$ require nonlinear optimization techniques.

The second component of spatial models is closely related to that of spaces. It is the distance between any two points. In network models, distances can be defined at will. In $\mathbb{R}^2$, one of the $L_k$ distance functions is usually chosen, i.e. the $L_k$ distance between $P_i = (x_i, y_i)$ and $P_j = (x_j, y_j)$ is defined as $d_{ij}^k = \left[ |x_i - x_j|^k + |y_i - y_j|^k \right]^{1/k}$, and similar in higher-dimensional real spaces. In physical location, the selection of a metric is usually farly straightforward. In nonphysical location, e.g. determining the distance between the most preferred point of a customer in some feature space (of a particular line of products, say) and the perceived

location of a product in that space, the choice of a metric is nontrivial. Here the $L_1$ distance corresponds to the philosophy that "differences (with respect to individual features of the product) build up", whereas the Chebyshev metric $L_\infty$ indicates that two points are considered just as different as their largest individual difference.

The third important component is that of customers. The term refers to standard physical location problems and it also holds for product selection models. In political models, customers correspond to voters and in work allocation problems the customers represent tasks. One crucial characteristic of customers is their distribution in space and another is their "purchasing" behavior. Except in network models, most authors assume that customers (or more accurately, purchasing power) is uniformly distributed in space. In reality, it is more likely that dense "customer clouds" (an analogy from $\mathbb{R}^3$) have formed around traditionally favored products in feature spaces, positions of lobbying groups in issue spaces or in more or less well-to-do parts of a city. Furthermore, purchasing behavior is an important factor. In a political election model, each voter can cast only zero or one votes irrespective of the location of the candidates. In a product selection model, a customer may very well be persuaded to purchase more of a product given that the price, quality, or other features are agreeable to him. In other words, the elasticity of customer demand needs to be specified.

The fourth component of spatial models is that of attraction functions. An attraction function measures the intensity by which a customer is attracted to a facility. Components of attraction functions are mostly facility-customer distance and sometimes weight. In general, the attraction

decreases as the distance gets larger and the attraction increases with increasing weight. The meaning of the weights depends again on the specific application: if the facilities are stores or shopping malls, weight may model the floor space or the variety (i.e. number) of products offered. In the political and product design models the weight could be used to measure the degree to which the candidate or product is known to the potential customers. Some of the more popular attraction functions $\Phi$ are the unweighted attractions with $\Phi = [\text{distance}]^{-1}$, the gravitational model $\Phi = [\text{weight}] / [\text{distance}]^2$ known for example from physics, a simplified version $\Phi = [\text{weight}] / [\text{distance}]$ and the linear function $\Phi = [\text{maximal attraction}] - [\text{attrition factor}] \cdot [\text{distance}]$.

The fifth component of spatial models consists of the facilities to be located. The number of facilities to be positioned by each of the participating players may be fixed or variable. As an example, one player may wish to locate an undecided number of branches of a supermarket chain in the given space. A related question is whether or not facilities engage in some kind of partial cooperation or if the setting is a purely competitive one.

The sixth component to be considered in competitive spatial models is that of the objectives of the facilities. Although profit maximization is a natural objective, other goals have important applications as well. For instance, the political model is of the voter (i.e. market share) maximization type. Specifically, in models where relocation or entry of new facilities takes place, facilities may wish to guard against other facilities invading "their" territory by positioning, so that their profit or market share is as large as possible given that another facility locates in the

worst possible spot (for the planning facility). This planning behavior corresponds to the standard minimax criterion in game theory. A completely different view could be taken by assuming central planning by an agency trying to maximize some welfare function. A possible objective could be to maximize the average service to customers or, the Rawlsian type of objective, to maximize the service to the least-served customer. Locational patterns and prices resulting from the optimization of welfare functions are known as social optima. Any combinations of the above objectives, social and competitive, can be used.

The seventh component specifies the market rules. They may indicate that the market is initially empty and all facilities locate simultaneously or they do so in a sequential manner. A similar situation occurs if the number of facilities does not change but they are in the process of relocating. Market rules may also regulate the conditions for entry of a new facility in the market as well as the exit of existing facilities. Yet another possibility is the conditional location of the medianoid and centroid type where one new facility is to located amidst already existing facilities in a competitive environment.

The eighth and last major descriptor of a spatial model indicates whether it is deterministic or probabilistic. In a deterministic model it is assumed that a customer patronizes exclusively that facility he is most attracted to. This behavioral assumption will result in market areas associated with each of the facilities which then become the basis for the determination of profit, market share, etc. On the other hand, it is possible to use the attraction of a customer to a facility in a probabilistic model. Here one may use the ratio of the attraction to, say the i-th, facility to

the sum of attractions as an indicator for the proportion of purchases a customer will make at the i-th facility. By definition, market areas no longer exist in probabilistic models. Naturally, probabilistic models are considerably more complex than deterministic models.

## 3. THE ORIGINAL HOTELLING MODEL: A CRITIQUE

In this section we first formally define competitive spatial models in general as well as an equilibrium condition. Then we describe the original model by Hotelling in some length as it is the basis of many other similar models. This is followed by a discusssion of some of the problems associated with the original solution of the model. In the remainder of the section we introduce a variety of other profit-maximization models on a linear market These models are closely related to the original one but have very different solutions, stressing the sensitivity of the model with respect to its input parameters.

Let there be n facilities with $x^k$ denoting some attribute (such as location or price) of the k-th facility. $\bar{x}^k$ indicates that this attribute is at least temporarily fixed. Denote by $F_i(\bar{x})$ the objective function for the i-th facility given fixed attributes $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ and let $X^i$ represent the set of feasible attributes for $x^i$. Then the optimization problem for the i-th facility is

$$(P_i): \text{ Maximize } z_i = F_i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{i-1}, x^i, \bar{x}^{i+1}, \ldots, \bar{x}^n)$$

$$\text{subject to } x^i \in X^i.$$

In a closed market, i.e. with no facility entering or leaving, a competitive model consists of the repeated optimization or problem $P_i$ for i = 1, ..., n. This set of n optimization problems is either solved simultaneously, one

problem for each planning facility, or sequentially. In the latter case the first facility optimizes on the basis that all other facilities as well as their respective features (such as prices or weights) remain unchanged. Once the optimization is complete the solution is "implemented" and the next facility optimizes on that basis. After all n facilities have optimized, one optimization round is over. After a number of optimization rounds it may happen that the locations, prices, or any other attribute of the facilities converge to an equilibrium. In particular, define a Nash equilibrium as a situation in which

$$\max_{x^i \in X^i} F_i(\bar{x}^1, \ldots, \bar{x}^{i-1}, x^i, \bar{x}^{i+1}, \ldots, \bar{x}^n) \leq F_i(\bar{x}^1, \ldots, \bar{x}^{i-1}, \bar{x}^i, \bar{x}^{i+1}, \ldots, \bar{x}^n)$$

$$\forall\ i = 1, \ldots, n.$$

In other words, a Nash equilibrium has been achieved if none of the facilities can improve its situation by changing its current attributes.

We are now able to describe the original Hotelling model. Suppose that the market consists of a line segment of length $\ell$ which, without loss of generality, we assume to be unity. Such a market is usually referred to as a linear market. Customers are uniformly distributed along the market and the demand elasticity is infinite, i.e. each customer buys a given number of units of the product, no matter what its price. For simplicity we assume the total customer demand to be one. Two facilities engaging in full competition are to be located on the market, so that the planner of each facility has two decision parameters: its location and its price. A customer will pruchase the product from the facility with the lowest delivered price, i.e. the mill price set by the facility plus transportation costs. The unit transportation costs are assumed to be linear and to simplify matters further, let production costs be zero. In this setting, Hotelling concluded

that the major force governing the process is a strong pull of each of the facilities towards the other. This argument is based on the gain a facility realizes between itself and its competitor and its own increasing hinterland by moving towards the other facility. This argument ignores the fact that facilities located very close to each other will engage in very active price competition as Bertrand [4] has noted. This is a counter force which keeps the two facilities apart.

The above argument is now formulated mathematically. In this we basically follow d'Aspremont, Gabszewicz and Thisse [7] . Suppose that the two facilities A and B locate a and b units from the left and right ends of the market, respectively, so that A is located to the left of B. The mill prices of the two facilities are denoted by $p_A$ and $p_B$ and the unit transportation costs are c. If the amrket extends from 0 to 1 then the price to be paid by a customer at some point $y \in [0; 1]$ is $p_A + cd_{yA}$ if the product is purchased fro, A and $p_B + cd_{yB}$ if it is bought from B, given that $d_{ij}$ denotes the distance between points i and j. As a customer will buy from the facility with the lowest delivered price, the price to be paid by any customer on the market with given facility locations of A and B is the lower envelope of the two V-shaped functions shown in Figure 1 by the dashed line. This also indicates that at point x, a customer is indifferent between the facilities A and B whereas for customers to the left of x, A is the preferred facility and B is preferred by customers located to the right of x. Thus A's market area is the line segment $\overline{0x}$ and B's market ares is $\overline{x1}$. If V(A) and V(B) denote the market (= Voronoi) areas of A and B, respectively, then their respective profits can be written as $\mathcal{P}_A = V(A) \cdot p_A$ and $\mathcal{P}_B = V(B) \cdot p_B$. Assuming that the boundary between the two market areas is at a distance of $\epsilon$ units to the right of A, we find that $\epsilon = (1/2c)[p_B - p_A + c(1-a-b)]$ , so

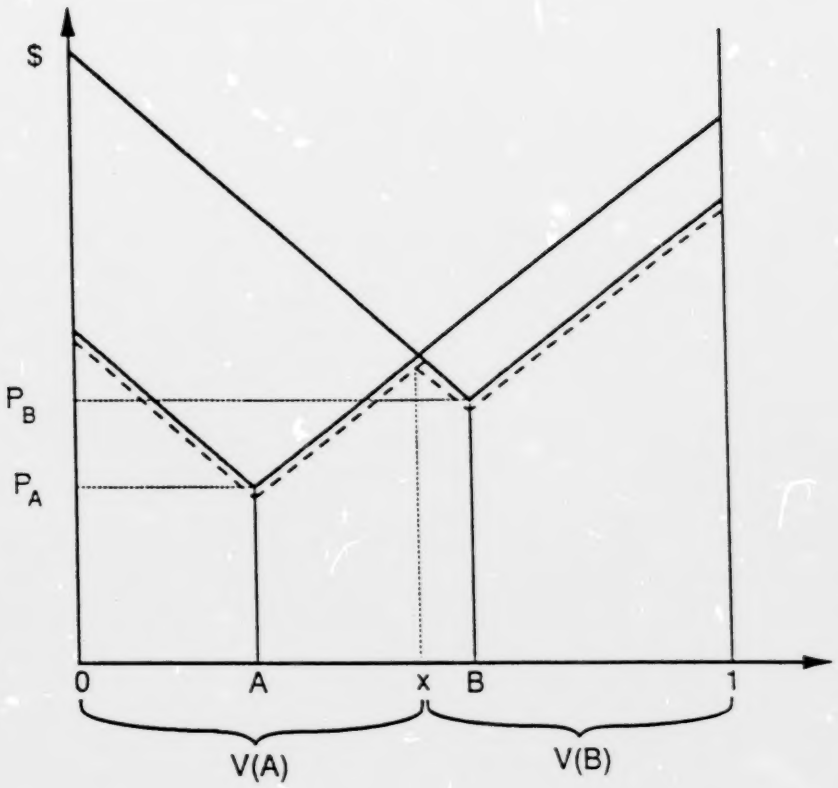


Figure 1: Market shares of two competing facilities

that $\mathcal{P}_A = ap_A + \frac{1}{2}(1-a-b) + (1/2c)p_A p_B - (1/2c)p_A^2$ and similarly for facility B. Partial differentiation with respecto to the prices, i.e. $\partial \mathcal{P}_A / \partial p_B = 0$ and $\partial \mathcal{P}_B / \partial p_B = 0$ results in two simultaneous linear equations whose solutions are the optimal prices $\bar{p}_A = c\left[1 + (a-b)/3\right]$ and $\bar{p}_B = c\left[1 - (a-b)/3\right]$. The optimal profits are then $\bar{\mathcal{P}}_A$ and $\bar{\mathcal{P}}_B$, respectively. As $\partial \bar{\mathcal{P}}_A / \partial a > 0$ and $\partial \bar{\mathcal{P}}_B / \partial b > 0$, the conclusion is to increase a and b, i.e. the two facilities should move towards each other. This is the original Hotelling result: both facilities cluster at the center. Referring to a one-dimensional feature space, this result is frequently called the principle of minimal differentiation. In contrast to it stands the social optimum of the same model: minimizing the sum of transportation costs leads to locations of the facilities at 1/4 and 3/4, respectively. Note that the competitive model assumes a two-tier process: for any pair of locations, the facilities are aware of the optimal prices. Given such information about the prices, the facilities will then locate.

One of the shortcomings of the above analysis was pointed out by d'Aspremont, Gabszewicz and Thisse. If the V-shaped price function associated with, say, facility A moves downward (which is achieved by decreasing $p_A$), the right side of the angle with vertex at $(A, p_A)$ will at some point coincide with the right side of facility B's price function. In this case, the lower envelope of the price functions is not unique and the customers in that area are indeterminate. If $p_A$ drops further (as shown in Figure 2), then B's market is reduced to zero and facility A serves all customers. This behavior causes jump discontinuities in the profit function and is responsible for the nonexistence of equilibria in the Hotelling model. Note that it is precisely in the presumed equilibrium, viz. when both facilities are clustered close together, that relatively small price changes in the vicinity of the discontinuity result in instability. The instability of the Hotelling model

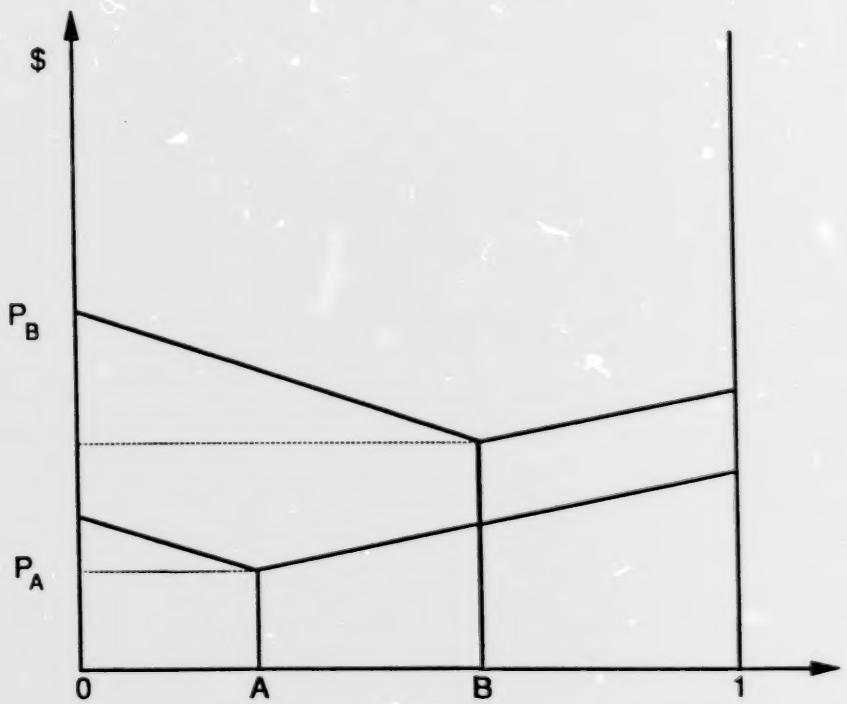


Figure 2: Price undercutting by facility A

is further amplified by the result by the aforementioned authors who have shown that the same model with quadratic rather than linear transportation costs does not only result in an equilibrium solution, but the locations at equilibrium are located at the respective ends of the market, i.e. maximal rather than minimal differentiation results.

Many variations of the original Hotelling model have been suggested. Some try to overcome the nonexistence of price equilibria for certain pairs of locations and others introduce different features so as to make the model more realistic. The following two sections include a number of these Hotelling-type models on a linear market. As in Anderson [1], we can distinguish between models with the following key features:

(a) equal mill prices for all facilities,

(b) different assumptions concerning consumer behavior,

(c) sequential entry and/or decision making,

(d) probabilistic models.

Clearly, some models belong to more than one of the above categories. The fourth section is devoted to models with features (b) - (d) whereas the fifth section consists entirely of market share models with assumption (a).

## 4. MAXIMUM PROFIT MODELS ON A LINEAR MARKET

One of the first responses to Hotelling's original work was by Lerner and Singer [21]. Their work falls into the same mold as does Smithies' [31]: in both papers the authors do away with the assumption of price inelasticity. In particular, Lerner and Singer assume a maximum price for all customers above which the demand for the product under consideration drops to zero. Some of the key issues of the Lerner and Singer study are summarized as follows.

Assuming that the decision maker at a facility considers the other's price and location as fixed, he optimizes his own price and location. Given the new situation, his opponent optimizes price and location. Repetition of this optimization procedure results in a number of price changes with periodical location changes. No equilibrium is reached. It is easy to imagine a number of ways a facility can react to its opponent's location and/or price changes. If a facility reacts only to price undercutting and it does so by relocating without, in turn, undercutting its opponent, then the authors show that a stable equilibrium exists for all values of the transportation costs.

In his study, Smithies assumes a linear price-quantity relation in order to establish the demand. He identifies two opposing forces required for an unclustered equilibrium: the possibility of gaining a larger market between the two facilities (which is achieved by moving closer towards the competitor) and the loss of markets in the uncontested hinterland of a facility (which results from moving away from some demand points). Disregarding differential prices, Hotelling's result of central clustering is obvious as his assumption of completely inelastic demand totally cancels the latter force. Moreover, Smithies does not accept the behavioral assumption of undercutting a competitor as this leads to economic war. He describes a number of scenarios ranging from full cooperation to full competition with respect to both prices and location.

Even the few examples cited above show the instability of the original Hotelling model: relatively small changes in the model assumptions result in dramatic changes in the outcome. Another example of this instability is provided by Prescott and Visscher [27]. In one of their examples, a continuous

profit function is used which results in the two facilities locating at the end points rather than the center of the market. The same result is again achieved with a model recently put forward by Neven [24] .Rather than assuming that customers patronize the facility closest to them, he introduces a utility function. A customer's utility is defined as a constant from which the squared customer-facility distance as well as the product price at that facility are deducted. If the fixed costs are zero, the two facilities enter the market sequentially and do not relocate, then maximal differentiation follows. Increasing the fixed costs makes the two facilities move closer together but they will not cluster at the center.

The concept of sequential market entry, used in the aforementioned study by Neven, is by no means new. It was most prominently used by Prescott and Visscher in an analysis reported below. Anderson's [1] study is based on Prescott and Visscher's analysis, except that he assumes the prices to be endogenous. Furthermore, he uses the well-known leader - follower concept first introduced by Stackelberg in the 1930s. Let one facility, say A, be the leader in the location game and suppose that B is the follower. On the other hand, facility B is assumed to be the leader in the second-stage price game where facility A is the follower. An equilibrium exists where A locates at the center of the market and B located very close to one of its ends. It turns out that A's profit is about twice as large as that of B. Yet another study which results in location patterns different from the supposed central clustering of Hotelling is that by Artle and Carruthers [2] . Here one extra stage is added to the process: a land owner rents locations on a linear market to facilities with the objective of maximizing his rent income. Clearly, the higher the profits of the facilities, the more rent can be extracted from the facilities, i.e. the land owner is interested in maximizing the sum of profits of the two facilities. Assuming that the price differences between any pair of

facilities are less than the transportation costs between those facilities and letting the land owner freely determine the number of facilities on the market, it turns out that two facilities will be allowed to locate. They will locate asymmetrically at 0.0 and 0.4 and they charge different prices. It should be pointed out that asymmetry was already observed by Hotelling in a related model including a land owner.

The recent report by Bonanno [5] follows a situation originally proposed by Teitz [32]. Rather than considering n facilities, each competing against the other, "players" are introduced and each player now operates a number of facilities. Here, one player is responsible for two facilities, the other for only one facility. Assuming quadratic transportation costs, fixed setup costs for facilities, an upper bound on the price a customer is willing to pay and let the first player enter the market as a monopolist and locate his two facilities. He will place them at 1/4 and 3/4, respectively. Given that situation, the second player now locates his single facility. He will do so in the same way a small market-share maximizing facility reacts to the existing location of a larger facility (details are discussed in the succeeding section): it will locate away from the large facility. Here, the single facility locates midway between the existing facilities at the center of the market.

The latest attempt to solve the original Hotelling model without changing the assumptions is that by Osborne and Pitchik [26]. As usual, the first stage consists of simultaneous choices of the facility locations and in the second stage, both facilities simultaneously choose their respective prices. It is shown that for certain pairs of locations, a unique pure price equilibrium exists, whereas for others, only mixed equilibria exist. For

the location game in the first stage a pure equilibrium exists with both facilities locating about 0.27 distance units away from the respective ends of the market. This is a remarkable result as such a competitive solution is very similar to the social optimum where both facilities are located 0.25 distance units away from the ends of the market.

The main purpose of this section was to demonstrate the sensitivity of the original Hotelling model by a variety of closely related models. Some further remarks to this effect can be found in the survey by Graitson [16] .

## 5. MAXIMUM MARKET SHARE MODELS ON A LINEAR MARKET

The models considered in this section have in common that the prices charged at all facilities are equal. In some cases such as gasoline and a number of brand-name items in grocery stores, this assumption may be approximately satisfied. But even if it is not, the loss of realism can be made good by incorporating various other features into the model. Due to the complexity of the problem this was not possible as long as prices were assumed to be variable. Our discussion commences with the simplest possible model which is the one where two profit maximizing facilities, i.e. duopolists, determine their respective locations. It should be reiterated that as long as the marginal costs do not exceed the fixed price, profit and revenue (and thus market share) optimization yields the same results.

It is apparent that two market share maximizing facilities will cluster at the center of the market. Any move away from the center will be countered by the other facility, so that the facility which first moved away from the center loses. Hence the principle of minimal differentiation holds in this case. Prescott and Visscher have shown that the same configuration results

when the duopolists enter the market sequentially and the first facility's objective is to locate in such a way that the second facility can do as little damage as possible to its market. This maximin objective is the same used in many game-theoretic models as well as in the design of search techniques where experiments are to be located optimally, e.g. the Fibonnacci search.

The above result does not however easily generalize to situations involving more than two facilities. It was already observed in the 1930s by Chamberlin [6] as well as Lerner and Singer [21] that there is no equilibrium in the case of three facilities. Depending on the sequence in which the facilities relocate and their original locations, one of two scenarios occurs. First, the facilities will cluster at the center which leaves one facility wedged in between two competitors. This facility has virtually a zero market share, a situation it tries to remedy by leapfrogging to the outside thus making another facility the interior one. Now the leapfrogging either continues closely around the center or, as Lerner and Singer described it, two groups of one and two facilities, respecitvely, expand outwards until they reach the first and third quartiles at which time they contract to an off-center group of three; subsequently the entire group moves slowly towards the center. As soon as that is reached, the same sequence of moves is repeated. Eaton and Lipsey [11] have set up necessary and sufficient conditions for the existence of equilibria given n facilities on a linear market. In particular, a configuration of locations is an equilibrium if and only if no facility has a market more than twice as large as another facility and the two peripheral facilities are paired. It can then be shown that only for n = 3 facilities, no equilibrium exists and for n > 5 facilities, an infinite number of equilibria exists. Shaked [30]

has shown that although the three-facility problem has no equilibrium in pure strategies, it does so when mixed strategies are considered. Assuming symmetry, each of the three facilities locates in the central half of the market with equal likelihood for all points and none of the facilities ever locates outside the first and third quartiles of the market. It is also interesting to note (particularly in view of Hotelling's original claim of wastefulness of the competitive solutions) that the total transportation costs for any given equilibrium are never more than twice those at the corresponding social optimum.

A different result for three facilities is obtained if the facilities enter the market sequentially with foresight, i.e. maximin objective. Prescott and Visscher show that in such a case two facilities locate at the first and third quartiles, respectively, and the third facility locates somewhere between the other two. Clearly, if relocation were allowed, leapfrogging would again occur and no equilibrium would be reached. Teitz's [32] model also involves three facilities, two (branches) belonging to one player, say A, with the other player, say B, operating the third facility. Using the usual market share maximization, the three facilities again leapfrog around the center of the market. With all relocations taking the same amount of time, player A expects 3/4 and B expects 1/4 of the market share. If both players operate multiple facilities each, an equilibrium is achieved if the player who operates more facilities locates them at the social optimal points and the other player optimizes on that basis. Here the "smaller" player, i.e. the one with less facilities, can be seen as safeguard, who assures "fair" service to the customers. Similar behavior occurs in the completely different context of location optimization in the presence of weights. Details are reported below.

Some further interesting results should be mentioned. Firstly, if facilities enter the market sequentially, the number of entering facilities is potentially infinite, a fixed location cost exists, then Prescott and Visscher have shown that -- assuming again that the facilities use foresight in their decisions -- all facilities except the last one locate in a non-agglomerate regular pattern.

For simplicity, consider for a moment the situation of two duopolists A and B, where facility B has already located at a point referred to as y. Then the market share function for A, located at x, is

$$V(x) = \begin{cases} x + \frac{1}{2}(y-x) & \text{if } x < y \\ \frac{1}{2} & \text{if } x = y \\ (1-x)\ \frac{1}{2}(x-y) & \text{if } x > y \end{cases}$$

which is piecewise linear but discontinuous. In order to overcome this discontinuity, Kohlberg [20] suggests introducing a behavioral component according to which customers do not minimize the distance they have to drive (i.e. they do not choose the closest facility) but minimize travel plus waiting time at the facility. The latter is assumed to be an increasing function of a facility's market share. The resulting market share function is continuous and it is hoped that this continuity adds an equilibrium to the case of $n = 3$ facilities. Surprisingly, it is proved that in this model no equilibrium exists for all $n > 2$.

Another attempt to overcome the instability of the three-facility model is that by dePalma, Ginsburgh and Thisse [8]. The model is probabilistic in nature with the probability of a customer patronizing a facility being a normalized exponential function of the attractiveness of the facility (a constant) which decreases with increasing customer-facility distance. In the normalization, a factor $\mu$ is used which symbolizes the heterogeneity

of customers' tastes. With c denoting the unit transportation costs, the ratio $(\mu/c)$ is examined. Clearly, $(\mu/c) = 0$ is the standard case described above in which no equilibrium exists. The authors of the study show that if $(\mu/c)$ is increased from 0, symmetric dispersed equilibria start to exist. Increasing $(\mu/c)$ further, both symmetric dispersed as well as agglomerated equilibria exist. If the ratio $(\mu/c)$ is increased even further, then only agglomerated equilibria exist. As an application, consider a one-dimensional product space where firms wish to locate products. If the customers are discriminating, then c is large and thus for any given $\mu$ , $(\mu/c)$ tends to be small. This results in rather different products, a situation which is typical for luxury products. On the other hand, if the decision makers of the facilities face non-discriminating customers, $(\mu/c)$ will be large and only agglomerated equilibria exist, i.e. the products tend to be very similar.

The introduction of weights into the model allows a different look at the problem. Suppose that customers are attracted to a facility according to an attraction function comprised of the ratio (facility weight)/(customer-facility distance). First of all, we observe that the resulting market areas need no longer be connected (or convex in $\mathbb{R}^d$ , $d \geq 2$). As an example, consider a linear market of unit length with one facility A of weight 3 located at 0.3 and a facility B of weight 1, located at 0.7. Then B's market area extends from 0.6 to 0.9 whereas A's market area consists of the area between 0.0 and 0.6 as well as that between 0.9 and 1.0. A linear-time procedure for the determination of such market areas has been presented by Aurenhammer and Edelsbrunner[3] .

Given an attraction function of the type (weight)/(distance)as described above, two possibilities arise: either (re-)optimize the weights or (re-)

optimize the locations. First consider the reoptimization of weights. For this purpose it is useful to define a Voronoi point as a point on a d-dimensional market where the attractions of at least (d+1) facilities are equal and no other facility is considered more attractive at that point. In the above numerical example, the points 0.6 and 0.9 are both Voronoi points. Eiselt and Laporte [13] have shown that the market size of a facility as a function of any of its weights is picewise concave. Discontinuities occur at weights where new market segments start to form (if the weight is increasing) or disappear (if the weight is decreasing). Assuming increasing weights, these new market segments form around the Voronoi points. Results based on repeated optimization with this model are not available.

Some numerical results concerning equilibria are however known for a simplified related model. Suppose that a facility's market area extends from that point to its left where its attraction equals that of its left neighbor, and similarly on its right. This model can be considered as a convexified version of the above model. Assuring a linear cost function and fixed locations, the weights are optimized for all facilities simultaneously. Then the results are announced and the weights are reoptimized at all facilities given the newest information concerning the weights of all competitors. It turns out that this repeated optimization procedure does converge towards equilibrium weights very quickly. In particular, within three "optimization rounds" the weights are within two percentage points of the equilibrium weights. Further details can be found in Eiselt, Laporte and Pederzoli [15] .

A few results for location optimization in the weighted case are also available. The optimal location of a new facility in the original nonconvex case has been discussed by Eiselt and Laporte [14] but again, no results concerning equilibria are available. An interesting result concerns the behavior of duopolists in the weighted model: given sequential reoptimization, i.e. alternating relocation of the facilities with fixed weights, it turns out that no equilibrium in the traditional sense is reached. The small facility, i.e. the one with the small weight, locates away from the large one which, in turn, locates "on top" of the small one. In this way they shuttle from one side of the market to the other. The process does however converge in the sense that the facilities do only jump between two different points which are both located at a distance from the respective ends of the market at a distance equal to the proportion of the smaller weight to the sum of both weights. Similar behavior can be observed in the case of two facilities with equal weights, one of which is a market share maximizer and the other minimizes the total customer travel distance, i.e. is a social optimizer. Again, repeated and sequential relocation yields no equilibrium but permanent shuttling between two points. It so happens that these two points are precisely the social optima, i.e. the first and third quartiles of the market. This means that half the time the social optimum is realized and the other half a noncentral agglomeration exists. In this way, one social optimizer can force an individual optimizer to realize social optima for at least part of the time.

## 6. MODELS IN GENERAL SPACES

The quest for more realistic models has made researchers look for spaces which resemble reality closer than a line segment. One natural

extension is a bounded subset of the Euclidean plane. Clearly, the difficulties encountered in the models based on line segments have by no means vanished, they are actually greatly amplified in models involving more general spaces. Consequently, many authors have relied on simulation, numerical methods, and approximation techniques rather than on analytical methods. Another promising extension is the use of networks.

First consider competitive location models in the plane. Recently, Hanjoul and Thill [17] have generated a number of different market areas based on a variety of transport cost functions. This work follows earlier studies of spheres of influence by market area theorists such as Huff. Eaton and Lipsey [11] discuss equilibrium problems on a disk for n = 3 to 17 market share maximizing facilities. They consider three patterns which are likely to represent equilibria. In the first, all facilities are located on a small circle around the center of the disk, the second pattern is basically the same as the first but with one facility at the center and the third is the Löschian hexagonal pattern with one facility at the center of the disk. Assuming sequential relocation in some fixed order, the authors compute the market shares of the facilities in case they relocate. It turns out that the facilities in the first pattern are located at local but not global maxima, so that they would relocate. The same holds true for the second pattern with the central facility presenting an additional problem (i.e. for n = 3 it is not even located at a local optimum). It is also shown that the Löschian pattern breaks up completely after only one optimization round. In summary, none of the proposed patterns represents an equilibrium: facilities tend to move close to one of their neighbors and pair up with them, a behavior previously observed on linear markets.

To better appreciate the complexity of the problem we will briefly analyze the required computations in some detail. First, for any given pattern of facilities, the market areas of all facilities are determined as functions of their current respective locations. Given the $L_2$ metric, the market areas are linearly bounded and convex polygons (the $L_2$ metric is actually the only one to generate market areas with such properties). These polygons are now triangulated and the resulting market area function can be optimized by the appropriate methods. It should be noted that the domain of this function is nonlinearly bounded and that only local optima may be achieved.

The first to use mathematical programming techniques for finding equilibria for a large number of facilities in the Euclidean plane are Okabe and Suzuki [25] . Their largest problem consists of 256 facilities located on a unit square. The authors first determine the social optimal pattern and it is no surprise that the result is an almost perfect honeycomb with only minor disturbances along the edges. Then the competitive model with sequential relocation is solved with the help of efficient methods for the determination of Voronoi diagrams. As the global optimization problem cannot be solved exactly, an approximation algorithm is employed. In particular, for the relocating facility the optimal location in the vicinity of its current location and that near the center of the largest empty circle (found with fast methods from computational geometry) are both determined. The better of the two locations is considered optimal and the facility is relocated accordingly. With random and social optimum patterns as respective initial solutions, a regular pattern emerges quickly and it stabilizes after about a hundred optimization rounds. Surprisingly (and contrary to the Eaton-Lipsey results reported above), a fairly regular pattern without pairing results which is not unlike the social optimal honeycomb. The competitive solution

does however exhibit a denser pattern of facilities along the edges of the unit square. This is quite unexpected as facilities along the border do not have any hinterland and thus a smaller market which usually indicates less facilities competing for the space. Even though the competitive solution is similar to the social optimum (a result resembling that by Osborne and Pitchik), the transformation of the latter to the former is not a short one. Actually, if the reoptimization process is initialized with the social optimum, this pattern is destroyed immediately and the procedure takes about as long to converge towards the competitive equilibrium as it does with a random distribution as initial pattern.

Finally, we should say a few words about competitive locations on networks. ReVelle [29] proposes a procedure which locates a given number of new facilities on a network. It is assumed that all facilities-to-be-located belong to one individual player, that the demands originate at the nodes of the network and that the facilities -- the existing ones as well as the new ones -- are also located at the nodes. The new facilities in this medianoid-type problem are located so as to maximize their collective market share. The recent work of Dobson and Karmarkar [9] deals with a similar problem. Various notions of stability are defined and some of their relations are explored. An NP-hard mathematical programming problem is set up which finds a number of locations for facilities, so that each facility is profitable but additional competitors would not make enough so that they will not locate and the established facilities are thus protected.

## 7. CONCLUSION

In this review we have outlined some of the problems arising in competitive spatial modeling. The field is growing at a fast pace and new research encompasses several areas. The different tools required for the analysis of these models, such as computational geometry and nonlinear programming as well as the multitude of applications of spatial models from politics to management and engineering are the major reasons for the true interdisciplinarity of the field.

As far as future research is concerned, various leads may be followed. On the one hand, further insight into network models as well as d-dimensional spaces with different metrics is essential for the understanding of the dynamics of the problem. Another possibility is the study of patterns which evolve if the facilities follow different objectives, e.g. profit maximizer, return-on-investment maximizer, social optimizer, etc. Work in this field could increase understanding of the effects of, say, a government facility competing with one or more free market facilities.

## REFERENCES


[ 1] S. Anderson, Spatial competition and price leadership, International Journal of Industrial Organization 5 (1987) 369-398.

[ 2] R. Artle and N. Carruthers, Location and market power: Hotelling revisited, Journal of Regional Science 28 (1988) 15 - 27.

[ 3] F. Aurenhammer and H. Edelsbrunner, An optimal algorithm for constructing the weighted Voronoi diagram in the plane, Pattern Recognition 17 (1984) 251 - 257.

[ 4] J. Bertrand, Théorie mathématique de la richesse sociale, Journal des Savants 48 (1883) 499 - 508.

[ 5] G. Bonanno, Location choice, product proliferation and entry deterrence, Review of Economic Studies 54 (1987) 37 - 45.

[ 6] E. Chamberlin, The Theory of Monopolistic Competition (Harvard University Press, Cambridge, 1933).

[ 7] C. D'Aspremont, J.J. Gabszewicz and J.-F. Thisse, On Hotelling's 'Stability in competition' , Econometrica 47 (1979) 1145 - 1150.

[ 8] A. De Palma, V. Ginsburgh and J.-F. Thisse, On existence of location equilibria in the 3-firm Hotelling problem, The Journal of Industrial Economics 36 (1987) 245 - 252.

[ 9] G. Dobson and U.S. Karmarkar, Competitive location on a network, Operations Research 35 (1987) 565 - 574.

[10] W. Domschke and A. Drexl, Location and Layout Planning: An International Bibliography, vol. 238 in Lecture Notes in Economics and Mathematical Systems (Springer, Berlin, 1985).

[11] B.C. Eaton and R.G. Lipsey, The principle of minimum differentiation reconsidered: some new developments in the theory of spatial competition, Review of Economic Studies 42 (1975) 27 - 49.

[12] H.A. Eiselt, Modeling business problems with Voronoi diagrams, working paper # 87-23 (revised version, May 1988), Faculty of Administration, University of New Brunswick, Fredericton, N.B., Canada.

[13] H.A. Eiselt and G. Laporte, Trading areas of facilities with different sizes, Recherche operationnelle/Operations Research 22 (1988) 33 - 44.

[14] H.A. Eiselt and G. Laporte, Location of a new facility on a linear market in the presence of weights, Asia-Pacific Journal of Operational Research (forthcoming, 1988).

[15] H.A. Eiselt, G. Laporte and G. Pederzoli, Optimal sizes of facilities on a linear market, Mathematical and Computer Modelling, (forthcoming, 1988).

[16] D. Graitson, Spatial competition à la Hotelling: a selective survey, The Journal of Industrial Economics 31 (1982) 15 - 25.

[17] P. Hanjoul and J.-C. Thill, Elements of planar analysis in spatial competition, Regional Science and Urban Economics 17 (1987) 423 - 439.

[18] H. Hotelling, Stability in competition, Economic Journal 39 (1929) 41 - 57.

[19] D.L. Huff, Defining and estimating a trading area, Journal of Marketing 28 (1964) 34 - 38.

[20] E. Kohlberg, Equilibrium store locations when consumers minimize travel time plus waiting time, Economics Letters 11 (1983) 211 - 216.

[21] A.P. Lerner and H.W. Singer, Some notes on duopoly and spatial competition, The Journal of Political Economy 45 (1937) 145 - 186.

[22] A. Lösch, Die räumliche Ordnung der Wirtschaft (G. Fischer, Jena, 1944).

[23] R.F. Love, J.G. Morris and G.O. Wesolowski, Facility Location: Models & Methods (North-Holland, New York, 1988).

[24] D.J. Neven, Endogenous sequential entry in a spatial model, International Journal of Industrial Organization 5 (1987) 419 - 434.

[25] A. Okabe and A. Suzuki, Stability of spatial competition for a large number of firms on a bounded two - dimensional space, Environment and Planning A 19 (1987) 1067 - 1082.

[26] M.J. Osborne and C. Pitchik, Equilibrium in Hotelling's model of spatial competition, Econometrica 55 (1987) 911 - 922.

[27] E.C. Prescott and M. Visscher, Sequential location among firms with foresight, The Bell Journal of Economics 8 (1977) 378 - 393.

[28] W.J. Reilly, Methods for Study of Retail Relationships, Research Monograph No. 4, Bureau of Business Research, The University of Texas, Austin, 1929.

[29] C. ReVelle, The maximum capture or 'sphere of influence' location problem: Hotelling revisited on a network, Journal of Regional Science 26 (1986) 343 - 358.

[30] A. Shaked, Existence and computation of mixed strategy Nash equilibrium for 3-firms location problem, The Journal of Industrial Economics 31 (1982) 93 - 96.

[31] A. Smithies, Optimum location in spatial competition, The Journal of Political Economics 49 (1941) 423 - 439.

[32] M.B. Teitz, Locational strategies for competitive systems, Journal of Regional Science 8 (1968) 135 - 148.

[33] A.H. Thiessen, Precipitation averages for large areas, Monthly Weather Review 39 (1911) 1082 - 1084.

[34] G. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques, deuxieme mémoire: recherches sur les paralleloèdres primitifs, Journal für reine und angewandte Mathematik 134 (1908) 198 - 287.

[35] R.E. Wendell and R.D. McKelvey, New perspectives in competitive location theory, European Journal of Operational Research 6 (1981) 174 - 182.